

**A VARIATIONAL FORMULATION FOR A TWO PHASE
MODEL IN SUPERCONDUCTIVITY AND A RESPECTIVE
NUMERICAL EXAMPLE**

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Abstract

This short communication develops a variational formulation for modeling a two phase model in superconductivity. The results are based on standard tools of calculus of variations and optimization theory. Moreover, we emphasize the context here addressed is essentially an Euler-Bernoullian one.

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1 Introduction

The first part of this article develops a variational formulation for a model in superconductivity.

Such results are based on tools of calculus of variations and related Lagrange multipliers approach. It is worth emphasizing, in the model here presented, a particle system corresponding to a fluid phase a , as submitted to a decreasing in temperature, partially changes into a super-conducting phase b . We emphasize, in such a here presented model, the super-conducting phase b behaves somewhat like a "single particle field", with approximately a unique quantum energy level.

Hence, concerning such scenarios, the particle mass constraint functionals for the phases a e b stand for

$$J_{Aux_1^a} = \int_0^{t_f} \int_{\Omega} E_1^a(y, t) \left(\int_{\Omega} (|\phi_a(x, y, t)|^2 dx - m_p^a) \right) dy dt, \quad (1)$$

where for such a phase a , the energy level $E_1^a(y, t)$ depends on y and t , and

$$J_{Aux_1^b} = \int_0^{t_f} E_1^b(t) \left(\int_{\Omega} (|\phi_b(x, t)|^2 dx - m_p^b) \right) dt, \quad (2)$$

where for such a phase b , the energy level $E_1^b(t)$ does not depend on y , that is, depends only on t , qualitatively behaving as a single energy state at each t .

Furthermore, we highlight the two main fields for such a system are a position one and a velocity one, which we specify below in the next lines.

It is worth mentioning similar models in superconductivity (the standard Ginzburg-Landau ones) are addressed in [1, 2].

As standard references in theoretical fluid mechanics we would cite [3, 4, 5, 6, 7]. For related numerical methods we would mention [8, 9, 10, 11]. Concerning other similar models, we would cite [12, 13, 14, 15, 16, 17].

Finally, details on the Sobolev spaces involved may be found in [18].

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and simply connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

Generically, we consider a electronic density motion initially modeled as a particle system one, through the field of position $\mathbf{r} : \Omega \times [0, t_f] \rightarrow \mathbb{R}^3$ and field of velocities $\mathbf{v}$ where $\mathbf{v}(\mathbf{r}(x, t), t)$ stands for the velocity at the position $\mathbf{r}(x, t)$ and time $t \in [0, t_f]$.

Moreover, the gravity field is denoted by

$$\mathbf{g} = -g\mathbf{k},$$

related to the $\mathbb{R}^3$ basis

$$\{\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0), \mathbf{k} = (0, 0, 1)\}.$$

2 The mathematical model

In this section, we develop in details the concerning variational formulation.

We emphasize such a set $\Omega \subset \mathbb{R}^3$ stands for a control volume, in which along a time interval $[0, t_f]$, where $t \in [0, t_f]$ denotes time, an electronic density develops a compressible motion.

For a particle system corresponding to a normal phase a , we start by defining the following electronic density field,

$$\hat{\rho}_a(x, y, t) = \left(|\phi_a(x, y, t)|^2 \frac{|\phi_a^M(y, t)|^2}{m_p^a} \right) \quad (3)$$

Here m_p^a denotes the mass of a single particle type a .

The macroscopic fluid density for such phase a , is defined by

$$\rho_a(y, t) = |\phi_a^M(y, t)|^2.$$

We recall that in the super-conducting phase b has approximately a unique energy level, which depends only on t but does not depend on the macroscopic variable y .

For such a particle system corresponding to the super-conducting phase b , we start by defining the following density field,

$$\hat{\rho}_b(x, y, t) = \left(|\phi_b(x, t)|^2 \frac{|\phi_b^M(y, t)|^2}{m_p^b} \right) \quad (4)$$

Here m_p^b denotes the mass of a single particle type b .

The macroscopic electronic fluid density for such phase b , is defined by

$$\rho_b(y, t) = |\phi_b^M(y, t)|^2.$$

It is worth highlighting that generically, we denote

$$L^2(\Omega \times [0, t_f]) = \left\{ h : \Omega \times [0, t_f] \rightarrow \mathbb{R} \text{ measurable} : \int_0^{t_f} \int_{\Omega} |h(x, t)|^2 dx dt < +\infty \right\}, \quad (5)$$

$$L^2(\Omega \times [0, t_f]; \mathbb{R}^3) = \left\{ h = (h_1, h_2, h_3) : \Omega \times [0, t_f] \rightarrow \mathbb{R}^3 : h_k \in L^2(\Omega \times [0, t_f]), \forall k \in \{1, 2, 3\} \right\}, \quad (6)$$

$$W^{1,2}(\Omega \times [0, t_f]) = \left\{ h : \Omega \times [0, t_f] \rightarrow \mathbb{R} : h \in L^2(\Omega \times [0, t_f]) \text{ and } h_t, h_{x_j} \in L^2(\Omega \times [0, t_f]), \forall j \in \{1, 2, 3\} \right\}, \quad (7)$$

where the partial derivatives h_t, h_{x_j} are understood in a distributional sense, $\forall j \in \{1, 2, 3\}$.

Moreover,

$$W^{1,2}(\Omega \times [0, t_f]; \mathbb{R}^3) = \left\{ h = (h_1, h_2, h_3) : \Omega \times [0, t_f] \rightarrow \mathbb{R}^3 : h_k \in W^{1,2}(\Omega \times [0, t_f]), \forall k \in \{1, 2, 3\} \right\}. \quad (8)$$

Similar notations are valid for other sets and other related complex and vectorial cases.

With such notations in mind, we define the concerning function spaces

$$\begin{aligned} V_1^a = & \left\{ \phi_a \in W^{1,2}(\Omega \times \Omega \times [0, t_f]; \mathbb{C}) : \nabla_x \phi_a \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \times \Omega \times [0, t_f], \right. \\ & \nabla_y \phi_a \cdot \mathbf{n} = 0 \text{ on } \Omega \times \partial\Omega \times [0, t_f], \frac{\partial \phi_a(x, y, 0)}{\partial t} = 0 \\ & \left. \text{and } \frac{\partial \phi_a(x, y, t_f)}{\partial t} = 0 \text{ in } \Omega \times \Omega \right\} \end{aligned} \quad (9)$$

and

$$\begin{aligned} V_2^a = & \left\{ \phi_a^M \in W^{1,2}(\Omega \times [0, t_f]; \mathbb{C}) : \phi_a^M = 0 \text{ on } \partial(\Omega_1 \cup \Omega_2) \times [0, t_f], \right. \\ & \left. \phi_a^M(z, t) = \phi_{in}(z, t) \text{ on } \partial\Omega_{in} \times [0, t_f] \text{ and } \phi_a^M(z, 0) = \phi_0 \text{ in } \Omega \right\}, \end{aligned} \quad (10)$$

where

$$\partial\Omega = \partial\Omega_{in} \cup \partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_{out}.$$

Moreover we suppose such a union is quasi-disjoint and through $\partial\Omega_{in}$ and $\partial\Omega_{out}$ are allowed the entering and exiting of fluid mass, respectively.

We define also,

$$V_1^b = \left\{ \phi_b \in W^{1,2}(\Omega \times [0, t_f]; \mathbb{C}) : \nabla_x \phi_b \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \times [0, t_f], \right. \\ \left. \frac{\partial \phi_b(x, 0)}{\partial t} = 0 \text{ and } \frac{\partial \phi_b(x, t_f)}{\partial t} = 0 \text{ in } \Omega \right\} \quad (11)$$

and

$$V_2^b = \left\{ \phi_b^M \in W^{1,2}(\Omega \times [0, t_f]; \mathbb{C}) : \phi_b^M = 0 \text{ on } \partial(\Omega_1 \cup \Omega_2) \times [0, t_f], \right. \\ \left. \phi_b^M(z, t) = \phi_{in}(z, t) \text{ on } \partial\Omega_{in} \times [0, t_f] \text{ and } \phi_b^M(z, 0) = \phi_0 \text{ in } \Omega \right\}, \quad (12)$$

Observe that, we may denote

$$\rho(y, t) = \rho_a(y, t) + \rho_b(y, t).$$

2.1 The position fields

For the phase a , we start by denoting the macroscopic electronic position field as

$$\mathbf{r}_a : \Omega \times [0, t_f] \rightarrow \mathbb{R}^3,$$

where

$$\mathbf{r}_a(x, t) = ((X_a)_1(x, t), (X_a)_2(x, t), (X_a)_3(x, t)).$$

In an internal variable and relativistic context, the particle position field for such a phase a , is defined by

$$\hat{\mathbf{r}}_a(x, t) = (\mathbf{r}_a(x, t), t) + \tilde{\mathbf{r}}_a^p(\mathbf{r}_a(x, t), t) = (\mathbf{r}_a(x, t), t) + (ct, \tilde{\mathbf{r}}_a(\mathbf{r}_a(x, t))).$$

Here $c > 0$ denotes the speed of light.

For the phase b , we start by denoting the macroscopic electronic position field as

$$\mathbf{r}_b : \Omega \times [0, t_f] \rightarrow \mathbb{R}^3,$$

where

$$\mathbf{r}_b(x, t) = ((X_b)_1(x, t), (X_b)_2(x, t), (X_b)_3(x, t)).$$

In an internal variable and relativistic context, the particle position field for such a phase a , is defined by

$$\hat{\mathbf{r}}_b(x, t) = (\mathbf{r}_b(x, t), t) + \tilde{\mathbf{r}}_b^p(\mathbf{r}_b(x, t), t) = (\mathbf{r}_b(x, t), t) + (ct, \tilde{\mathbf{r}}_b(\mathbf{r}_b(x, t))).$$

We define also the following function space

$$\begin{aligned} V_3^a &= \{\mathbf{r}_a \in W^{1,2}(\Omega \times [0, t_f]; \mathbb{R}^3) : \mathbf{r}_a = \mathbf{0} \text{ on } (\partial\Omega_1 \cap \partial\Omega_2) \times [0, t_f] \\ &\text{and } \mathbf{r}_a(z, 0) = \mathbf{r}_0 \text{ in } \Omega\}, \end{aligned} \quad (13)$$

and assume

$$\tilde{\mathbf{r}}_a \in V_4 = W^{1,2}(\mathbb{R}^3; \mathbb{R}^3).$$

For the phase b , we define the following corresponding space

$$\begin{aligned} V_3^b &= \{\mathbf{r}_b \in W^{1,2}(\Omega \times [0, t_f]; \mathbb{R}^3) : \mathbf{r}_b = \mathbf{0} \text{ on } (\partial\Omega_1 \cap \partial\Omega_2) \times [0, t_f] \\ &\text{and } \mathbf{r}_b(z, 0) = \mathbf{r}_0 \text{ in } \Omega\}, \end{aligned} \quad (14)$$

and assume

$$\tilde{\mathbf{r}}_b \in V_4 = W^{1,2}(\mathbb{R}^3; \mathbb{R}^3).$$

2.2 About the system constraints

Denoting by $m_T = m_a(0) + m_b(0)$ the total system mass at $t = 0$, we have the following constraint

$$\begin{aligned} m_T &= m_a(0) + m_b(0) \\ &= m_a(t) + \int_0^t \int_{\partial\Omega} \rho_a(y, \tau) \mathbf{v}_a \cdot \mathbf{n} \, dS \, d\tau \\ &\quad + m_b(t) + \int_0^t \int_{\partial\Omega} \rho_b(y, \tau) \mathbf{v}_b \cdot \mathbf{n} \, dS \, d\tau \end{aligned} \quad (15)$$

where

$$\begin{aligned} \mathbf{v}_a &= \mathbf{v}_a(\mathbf{r}_a(x, t), t), \\ \mathbf{v}_b &= \mathbf{v}_b(\mathbf{r}_b(x, t), t). \end{aligned}$$

and where

$$m_a(t) = \int_{\Omega} \rho_a(y, t) \, dt$$

and

$$m_b(t) = \int_{\Omega} \rho_b(y, t) \, dt,$$

on $[0, t_f]$.

Here $\mathbf{n}$ denotes outward normal field to $\partial\Omega = S$.

Relating the internal kinetics energy, in a relativistic context, we define the following functional

$$\begin{aligned}
& E_{C_1}^a \\
&= -\frac{1}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} \frac{\hat{\rho}_a(x, y, t)}{\sqrt{1 - \frac{\tilde{v}_a^2}{c^2}}} \frac{d\tilde{\mathbf{r}}_a^p}{dt} \cdot \frac{d\tilde{\mathbf{r}}_a^p}{dt} \sqrt{-g^a} |\det X'_a| dx dy dt \\
&= \frac{1}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} \frac{\hat{\rho}_a(x, y, t)}{\sqrt{1 - \frac{\tilde{v}_a^2}{c^2}}} (c^2 - \tilde{v}_a^2) \sqrt{-g^a} |\det X'_a| dx dy dt \\
&= \frac{1}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} c \hat{\rho}_a(x, y, t) \sqrt{c^2 - \tilde{v}_a^2} \sqrt{-g^a} |\det X'_a| dx dy dt \\
&= \frac{1}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} c \hat{\rho}_a(x, y, t) \sqrt{c^2 - \hat{g}_{jk}^a \frac{\partial(X_a)_j}{\partial t} \frac{\partial(X_a)_k}{\partial t}} \sqrt{-g^a} |\det X'_a| dx dy dt \\
&= \frac{1}{2} \int_0^{t_f} \int_{\Omega} e_C^a dy dt, \tag{16}
\end{aligned}$$

where generically, we have denoted, $(X_a)_0 = t$,

$$\begin{aligned}
g_{jk}^a &= \frac{\partial \tilde{\mathbf{r}}_a^p}{\partial (X_a)_j} \cdot \frac{\partial \tilde{\mathbf{r}}_a^p}{\partial (X_a)_k}, \forall j, k \in \{0, 1, 2, 3\} \\
\{\hat{g}_{jk}^a\} &= \{g_{jk}^a\}_{(j,k \in \{1,2,3\})},
\end{aligned}$$

$g^a = \det\{g_{jk}^a\}$, where

$$w \cdot v = -w_0 v_0 + \sum_{k=1}^3 w_k v_k$$

for

$$w = (w_0, w_1, w_2, w_3)$$

and

$$v = (v_0, v_1, v_2, v_3) \in \mathbb{R}^4,$$

and

$$e_C^a = \int_{\Omega} c \hat{\rho}_a(x, y, t) \sqrt{c^2 - \hat{g}_{jk}^a \frac{\partial(X_a)_j}{\partial t} \frac{\partial(X_a)_k}{\partial t}} \sqrt{-g^a} |\det X'_a| dx.$$

With analogous notation, the corresponding functional for the phase b is defined by

$$\begin{aligned}
& E_{C_1}^b \\
&= \frac{1}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} c \hat{\rho}_b(x, y, t) \sqrt{c^2 - \hat{g}_{jk}^b \frac{\partial(X_b)_j}{\partial t} \frac{\partial(X_b)_k}{\partial t}} \sqrt{-g^b} |\det X'_b| dx dy dt \\
&= \frac{1}{2} \int_0^{t_f} \int_{\Omega} e_C^b dy dt, \tag{17}
\end{aligned}$$

where,

$$e_C^b = \int_{\Omega} c \hat{\rho}_b(x, y, t) \sqrt{c^2 - \hat{g}_{jk}^b \frac{\partial (X_b)_j}{\partial t} \frac{\partial (X_b)_k}{\partial t}} \sqrt{-g^b} |\det X'_b| dx.$$

For the interacting and self-interacting energies, in a non-linear Schrödinger equation context, we define the following functional,

$$\begin{aligned} E_{in} = & \frac{\alpha_1^a}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} \rho_a(y, t)^2 dy dt \\ & - \frac{\beta_1^a}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} \rho_a(y, t) dy dt \\ & + \frac{\alpha_1^b}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} \rho_b(y, t)^2 dy dt \\ & - \frac{\beta_1^b}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} \rho_b(y, t) dy dt \end{aligned} \quad (18)$$

for appropriate $\alpha_1^a > 0$, α_1^b and $\beta_1^a, \beta_1^b > 0$.

On the other hand, we define

$$\begin{aligned} E_M = & \frac{1}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} \left(\frac{\gamma_1^a}{m_p^a} |\nabla \phi_a(x, y, t) - i w_a \mathbf{A} \phi_a(x, y, t)|^2 \right) |\phi_a^M(y, t)|^2 dx dy dt \\ = & \frac{1}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} \left(\frac{\gamma_1^b}{m_p^b} |\nabla \phi_b(x, t) - i w_b \mathbf{A} \phi_b(x, t)|^2 \right) |\phi_b^M(y, t)|^2 dx dy dt, \end{aligned} \quad (19)$$

for appropriate constants $\gamma_1^a, \gamma_1^b > 0$ and $w_a, w_b \in \mathbb{R}$.

Concerning the restrictions, already including the respective Lagrange multipliers, we define also the following functionals,

$$\begin{aligned} J_{Aux_1^a} = & \int_0^{t_f} \int_{\Omega} E_1^a(y, t) \left(\int_{\Omega} (|\phi_a(x, y, t)|^2 dx - m_p^a) \right) dy dt, \end{aligned} \quad (20)$$

$$\begin{aligned} J_{Aux_1^b} = & \int_0^{t_f} E_1^b(t) \left(\int_{\Omega} (|\phi_b(x, t)|^2 dx - m_p^b) \right) dt, \end{aligned} \quad (21)$$

$$J_{Aux_2} = \int_0^{t_f} E_2(t) \left(m_T - \int_{\Omega} \rho_a(y, t) dy - \int_0^t \int_{\partial\Omega} \rho_a(y, \tau) \mathbf{v}_a \cdot \mathbf{n} dS d\tau \right. \\ \left. - \int_{\Omega} \rho_b(y, t) dy - \int_0^t \int_{\partial\Omega} \rho_b(y, \tau) \mathbf{v}_b \cdot \mathbf{n} dS d\tau \right) dt. \quad (22)$$

$$J_{Aux_3^a} = \int_0^{t_f} \int_{\Omega} E_3^a(x, t) \left(\mathbf{v}_a(\mathbf{r}_a(x, t), t) - \frac{\partial \mathbf{r}_a(x, t)}{\partial t} \right) dx dt, \\ J_{Aux_3^b} = \int_0^{t_f} \int_{\Omega} E_3^b(x, t) \left(\mathbf{v}_b(\mathbf{r}_b(x, t), t) - \frac{\partial \mathbf{r}_b(x, t)}{\partial t} \right) dx dt,$$

$$J_{Aux_4} = - \int_0^T \int_{\Omega} \hat{P} \left(\frac{d\rho}{dt} + \rho_a \operatorname{div} \mathbf{v}_a + \rho_b \operatorname{div} \mathbf{v}_b \right) dy dt, \quad (23)$$

Moreover, considering the pressure and temperature scalar fields, we define

$$P_a = \frac{d(\hat{P}\rho_a)}{dt}, \\ P_b = \frac{d(\hat{P}\rho_b)}{dt},$$

where

$$\hat{P} \in V_5$$

$$V_5 = \{ \hat{P} \in W^{1,2}(\Omega \times [0, t_f]) : P = P_a + P_b = P(\hat{P}, \rho) = P_0, \text{ on } \partial\Omega \times [0, t_f] \}.$$

and the temperature fields, T_a, T_b through the equations

$$(C_v)_a \rho_a T_a = e_C^a \quad (24)$$

so that

$$(C_v)_b \rho_b T_b = e_C^b \quad (25)$$

for an appropriate real constant $C_v > 0$.

At this point, we define the entropy differentials

$$dS_a = -\hat{K}_a \frac{\rho_a}{\rho} \ln \left(\frac{\rho_a}{e \rho} \right) dy dt,$$

$$dS_b = -\hat{K}_b \frac{\rho_b}{\rho} \ln \left(\frac{\rho_b}{e \rho} \right) dy dt,$$

and the functional

$$E^S = \int_0^{t_f} \int_{\Omega} (T_a dS_a + T_b dS_b).$$

Moreover, we also define

$$e^S = T_a S_a + T_b S_b,$$

where

$$S_a = -\hat{K}_a \frac{\rho_a}{\rho} \ln \left(\frac{\rho_a}{e \rho} \right)$$

and

$$S_b = -\hat{K}_b \frac{\rho_b}{\rho} \ln \left(\frac{\rho_b}{e \rho} \right).$$

With such results in mind, neglecting the effect of electromagnetic fields on the temperature ones, we define the restriction

$$\begin{aligned} & (C_v)_a \rho_a \frac{dT_a}{dt} + (C_v)_b \rho_b \frac{dT_b}{dt} \\ &= K_a \nabla^2 T_a + K_b \nabla^2 T_b \\ & \quad - P_a \operatorname{div} \mathbf{v}_a - P_b \operatorname{div} \mathbf{v}_b + \frac{de^S}{dt}, \end{aligned} \tag{26}$$

for an appropriate real constants $K_a > 0$, $K_b > 0$.

At this point, we define the following function space

$$V_6^a = \{T_a \in W^{1,2}(\Omega \times [0, t_f]) : T_a(z, t) = T_1^a(z, t) \text{ on } \partial\Omega \times [0, t_f] \text{ and } T_a(z, 0) = T_0^a(z) \text{ in } \Omega\},$$

$$V_6^b = \{T_b \in W^{1,2}(\Omega \times [0, t_f]) : T_b(z, t) = T_1^b(z, t) \text{ on } \partial\Omega \times [0, t_f] \text{ and } T_b(z, 0) = T_0^b(z) \text{ in } \Omega\},$$

and

$$\begin{aligned} V_7^a &= \{\mathbf{v}_a \in W^{1,2}(\mathbb{R}^3; \mathbb{R}^3) : \mathbf{v}_a = \mathbf{0} \text{ on } (\partial\Omega_1 \cup \partial\Omega_2) \times [0, t_f] \\ & \quad \text{and } \mathbf{v}_a(z, 0) = \mathbf{v}_0^a \text{ on } \Omega\}. \end{aligned} \tag{27}$$

$$\begin{aligned} V_7^b &= \{\mathbf{v}_b \in W^{1,2}(\mathbb{R}^3; \mathbb{R}^3) : \mathbf{v}_b = \mathbf{0} \text{ on } (\partial\Omega_1 \cup \partial\Omega_2) \times [0, t_f] \\ & \quad \text{and } \mathbf{v}_b(z, 0) = \mathbf{v}_0^b \text{ on } \Omega\}. \end{aligned} \tag{28}$$

With such restrictions and definitions in mind, we define also the following functionals,

$$J_{Aux_5^a} = \int_0^{t_f} \int_{\Omega} E_5^a(y, t) ((C_v)_a \rho_a T_a - e_C^a) dy dt \quad (29)$$

$$J_{Aux_5^b} = \int_0^{t_f} \int_{\Omega} E_5^b(y, t) ((C_v)_b \rho_b T_b - e_C^b) dy dt \quad (30)$$

and

$$\begin{aligned} J_{Aux_6} = & \int_0^{t_f} \int_{\Omega} E_6(y, t) \left((C_v)_a \rho_a \frac{dT_a}{dt} + (C_v)_b \rho_b \frac{dT_b}{dt} \right. \\ & - K_a \nabla^2 T_a - K_b \nabla^2 T_b \\ & \left. + P_a \operatorname{div} \mathbf{v}_a + P_b \operatorname{div} \mathbf{v}_b - \frac{de^S}{dt} \right) dy dt. \end{aligned} \quad (31)$$

Finally, we define the following functional,

$$\begin{aligned} & J_{Aux_7^a} \\ = & \int_0^{t_f} \int_{\Omega} E_{50}^a(y, t) \left(\rho_a(y, t) - \int_{\Omega} \left(\frac{|\phi_a(x, y, t)|^2}{m_p^a} \right) dx |\phi_a^M(y, t)|^2 \right) dy dt \end{aligned} \quad (32)$$

$$\begin{aligned} & J_{Aux_7^b} \\ = & \int_0^{t_f} \int_{\Omega} E_{50}^b(y, t) \left(\rho_b(y, t) - \int_{\Omega} \left(\frac{|\phi_b(x, y, t)|^2}{m_p^b} \right) dx |\phi_b^M(y, t)|^2 \right) dy dt \end{aligned} \quad (33)$$

3 The main functional

Here we define the functional J , by

$$\begin{aligned} J = & \frac{1}{2} \int_0^{t_f} \int_{\Omega} \rho_a \mathbf{v}_a \cdot \mathbf{v}_a dy dt - \int_0^{t_f} \int_{\Omega} \rho_a \mathbf{g} \cdot \mathbf{r}_a dy dt \\ & + \frac{1}{2} \int_0^{t_f} \int_{\Omega} \rho_b \mathbf{v}_b \cdot \mathbf{v}_b dy dt - \int_0^{t_f} \int_{\Omega} \rho_b \mathbf{g} \cdot \mathbf{r}_b dy dt \\ & - K_1 \int_0^{t_f} \int_{\Omega} \rho_a \mathbf{A} \cdot \frac{\partial \mathbf{r}_a}{\partial t} dx dt \\ & - K_1 \int_0^{t_f} \int_{\Omega} \rho_b \mathbf{A} \cdot \frac{\partial \mathbf{r}_b}{\partial t} dx dt, \end{aligned} \quad (34)$$

for an appropriate real constant K_1 .

Moreover, for $\varepsilon > 0$ small, we define the functional J_1 , by

$$\begin{aligned}
 J_1 = & - \int_0^{t_f} \int_{\Omega} V(x, t) K_1(\rho_a + \rho_b) dx dt \\
 & + \frac{1}{8\pi} \int_0^{t_f} \int_{\Omega} |\text{Curl } \mathbf{A} - \mathbf{B}_0|_{\mathbb{R}^3}^2 dx dt \\
 & - \frac{1}{8\pi} \int_0^{t_f} \int_{\Omega} \varepsilon_0(\rho(\mathbf{r}_a, \mathbf{r}_b), \mathbf{r}_a, \mathbf{r}_b) \left| \nabla V + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right|_{\mathbb{R}^3}^2 dx dt \\
 & + \int_0^{t_f} \int_0^{t_f} \rho_a \mathbf{B} \times \frac{\partial \mathbf{r}_a}{\partial t} \cdot (\mathbf{r}_a(x, \varepsilon t) - \mathbf{r}_a(x, 0)) dx dt \\
 & + \int_0^{t_f} \int_0^{t_f} \rho_b \mathbf{B} \times \frac{\partial \mathbf{r}_b}{\partial t} \cdot (\mathbf{r}_b(x, \varepsilon t) - \mathbf{r}_b(x, 0)) dx dt, \tag{35}
 \end{aligned}$$

where $c > 0$ in atomic units denotes the speed of light,

$$\mathbf{B} = \mathbf{B}_0 - \text{Curl } \mathbf{A},$$

denotes the total magnetic field, $\mathbf{A}$ denotes a magnetic potential and $\mathbf{B}_0$ stands for an external magnetic field.

Furthermore, V is an external electric potential and

$$\mathbf{E} = \left(\nabla V + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right),$$

is the electric field resulting from the charged particles.

Defining

$$(V, \mathbf{A}) \in V_8 = W^{1,2}(\Omega \times [0, t_f]) \times W^{1,2}(\Omega \times [0, t_f]; \mathbb{R}^3)$$

and

$$\tilde{V} = V_1^a \times V_1^b \times V_2^a \times V_2^b \times V_3^a \times V_3^b \times V_5 \times V_6^a \times V_6^b \times V_7^a \times V_7^b \times V_8,$$

the final variational formulation $J_{total} : \tilde{V} \rightarrow \mathbb{R}$ stands for

$$\begin{aligned}
 J_{total} = & -J + J_1 \\
 & E_{C_1}^a + E_{C_1}^b + E_{in} \\
 & + E_M + J_{Aux_1^a} + J_{Aux_1^b} + J_{Aux_2} + J_{Aux_3^a} + J_{Aux_3^b} \\
 & J_{Aux_4} + J_{Aux_5^a} + J_{Aux_5^b} \\
 & + J_{Aux_6} + J_{Aux_7^a} + J_{Aux_7^b} - E^S. \tag{36}
 \end{aligned}$$

The variation of J_{total} in $\mathbf{r}_a$ which lead us to the correct Euler-Lagrange equation is defined by

$$\begin{aligned} & \hat{\delta}(J_{total})_{\mathbf{r}_a}(\mathbf{r}_a, \mathbf{r}_a(x, \varepsilon t), \dots, \mathbf{A}, V; \varphi) \\ &= \lim_{h \rightarrow 0} [J_{total}(\mathbf{r}_a + h\varphi, \mathbf{r}_a(x, \varepsilon t) + h\varphi, \dots, \mathbf{A}, V) \\ & \quad - J_{total}(\mathbf{r}_a, \mathbf{r}_a(x, \varepsilon t), \dots, \mathbf{A}, V)] / h \\ &= \mathbf{0}, \forall \varphi \in C_c^\infty(\Omega \times [0, T]; \mathbb{R}^3). \end{aligned} \quad (37)$$

We highlight the solution of the concerned system including such an Euler-Lagrange equation is computable, through the Newton's method for example, by expanding J_{total} as a quadratic approximate function in φ about $(\mathbf{r}_a(x, t), \mathbf{r}_a(x, \varepsilon t))$ in each iteration.

A similar remark is valid for the variation of J_{total} in $\mathbf{r}_b$.

Remark 3.1. *Generically denoting*

$$V = V(x, t) = \hat{V}(\mathbf{r}_a, \mathbf{r}_b),$$

the variation of J_{total} in $\mathbf{A}$ stands for

$$\begin{aligned} & \text{curl} (\text{curl } \mathbf{A} - \mathbf{B}_0) + \frac{1}{c} \frac{\partial}{\partial t} \left(\varepsilon_0 \left(\nabla \hat{V}(\mathbf{r}_a, \mathbf{r}_b) + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) \right) \\ & + \mathcal{O}(\varepsilon) + 4\pi K_1 \rho_a \frac{\partial \mathbf{r}_a}{\partial t} + 4\pi K_1 \rho_b \frac{\partial \mathbf{r}_b}{\partial t} \\ &= \mathbf{0}. \end{aligned} \quad (38)$$

Letting $\varepsilon \rightarrow 0$, we have the following Maxwell equation

$$\text{curl } \mathbf{B} - \frac{1}{c} \frac{\partial(\varepsilon_0 \mathbf{E})}{\partial t} - 4\pi \tilde{\mathbf{J}} = \mathbf{0},$$

where

$$\tilde{\mathbf{J}} = K_1 \rho_a \frac{\partial \mathbf{r}_a}{\partial t} + K_1 \rho_b \frac{\partial \mathbf{r}_b}{\partial t} \quad (39)$$

The variation of J_{total} in V stands for

$$-4\pi K_1(\rho_a + \rho_b) + \text{div} \left(\varepsilon_0 \left(\nabla \hat{V}(\mathbf{r}_a, \mathbf{r}_b) + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) \right) = \mathbf{0},$$

so that we obtain another Maxwell equation

$$\text{div} (\varepsilon_0 \mathbf{E}) = 4\pi K_1(\rho_a + \rho_b).$$

Moreover, from

$$\mathbf{B} = - \operatorname{curl} \mathbf{A} + \mathbf{B}_0$$

and assuming $\operatorname{div} (\mathbf{B}_0) = 0$, we have got

$$\operatorname{div} \mathbf{B} = - \operatorname{div} (\operatorname{curl} \mathbf{A}) = 0,$$

which is a third Maxwell equation.

Finally, the fourth Maxwell equation is obtained recalling that

$$\mathbf{E} = \nabla \hat{V}(\mathbf{r}_a, \mathbf{r}_b) + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t},$$

so that

$$\operatorname{curl} \mathbf{E} = \operatorname{curl} (\nabla V) + \frac{1}{c} \operatorname{curl} \left(\frac{\partial \mathbf{A}}{\partial t} \right),$$

and therefore,

$$\operatorname{curl} \mathbf{E} - \frac{1}{c} \left(\frac{\partial \operatorname{curl} \mathbf{A}}{\partial t} \right) = 0,$$

that is,

$$\operatorname{curl} \mathbf{E} - \frac{1}{c} \left(\frac{\partial (-\mathbf{B} + \mathbf{B}_0)}{\partial t} \right) = 0.$$

Thus, if $\mathbf{B}_0$ is time independent, that is,

$$\frac{\partial \mathbf{B}_0}{\partial t} = 0,$$

we have

$$\operatorname{curl} \mathbf{E} + \frac{1}{c} \left(\frac{\partial \mathbf{B}}{\partial t} \right) = 0.$$

In summary, we have got all the four Maxwell equations as necessary conditions for a extremal point of J_{total} .

The objective of this section is complete.

4 A numerical example

In this section, we develop a numerical example which models the phase transition of a super-conducting straight bar corresponding to a domain $\Omega = [0, 1] \subset \mathbb{R}$, from a normal phase a to a superconductive phase b .

Here we highlight for this one dimensional example we have generically denoted

$$\nabla \phi_a = \frac{d\phi_a}{dx}$$

and

$$\nabla^2 \phi_a = \frac{d^2 \phi_a}{dx^2}.$$

For such a model, already including a Lagrange multiplier E related to the mass constraint

$$\int_{\Omega} (|\phi_a|^2 + |\phi_b|^2) dx = 1,$$

we define the simplified functional $J : V_1 \rightarrow \mathbb{R}$, where

$$\begin{aligned} J(\phi_a, \phi_b, \mathbf{r}_a, \mathbf{r}_b, E) = & -\frac{1}{2} \int_0^{t_f} \int_{\Omega} |\phi_a|^2 \frac{\partial \mathbf{r}_a}{\partial t} \cdot \frac{\partial \mathbf{r}_a^*}{\partial t} dx dt \\ & + \frac{\gamma_1}{2} \int_{\Omega} \nabla \phi_a \cdot \nabla \phi_a dx + \frac{\alpha_1}{4} \int_{\Omega} |\phi_a|^4 dx \\ & - \frac{\beta_1}{2} \int_{\Omega} |\phi_a|^2 dx \\ & - \frac{1}{2} \int_0^{t_f} \int_{\Omega} |\phi_b|^2 \frac{\partial \mathbf{r}_b}{\partial t} \cdot \frac{\partial \mathbf{r}_b^*}{\partial t} dx dt \\ & + \frac{\gamma_1}{2} \int_{\Omega} \nabla \phi_b \cdot \nabla \phi_b dx + \frac{\alpha_1}{4} \int_{\Omega} |\phi_b|^4 dx \\ & - \frac{\beta_1}{2} \int_{\Omega} |\phi_b|^2 dx \\ & - \frac{E}{2} \left(\int_{\Omega} (|\phi_a|^2 + |\phi_b|^2) dx - 1 \right). \end{aligned} \quad (40)$$

where

$$V_1 = W_0^{1,2}(\Omega; \mathbb{C}^2) \times W^{1,2}(\Omega \times [0, t_f]; \mathbb{R}^2) \times \mathbb{R}.$$

In such a numerical example, we set, for the normal phase, $\gamma_1 = 1$, $\alpha_1 = 1000$ (higher self-interacting level), and initially $\beta_1 = \beta_1(T) = 1050$, where T denotes the sample Ω temperature.

For the super-conducting phase b , we define $\gamma_2 = 1$, $\alpha_2 = 1.0$ (much lower self-interacting level) and $\hat{\beta}_2 = \hat{\beta}_2(T) = 60$, where $\hat{\beta}_2$ will be specified in the next lines.

Setting

$$\begin{aligned} \mathbf{r}_a(x, t) &= \mathbf{C}_a e^{i w_a t} \\ \mathbf{r}_b(x, t) &= \mathbf{C}_b e^{i w_b t} \end{aligned}$$

where $i \in \mathbb{C}$ denotes the imaginary unit and $\mathbf{C}_a$ and $\mathbf{C}_b$ are appropriate vectors.

We also assume

$$\mathbf{C}_a \approx \mathbf{0}.$$

From the variation of J in ϕ_a , we obtain the following Euler-Lagrange equation,

$$\begin{aligned} & -\phi_a |\mathbf{C}_a|^2 w_a^2 - \gamma_1 \nabla^2 \phi_a + \alpha_1 |\phi_a|^2 \phi_a - \beta_1 \phi_a - E \phi_a \\ \approx & -\gamma_1 \nabla^2 \phi_a + \alpha_1 |\phi_a|^2 \phi_a - \beta_1 \phi_a - E \phi_a \\ \approx & 0. \end{aligned} \quad (41)$$

Moreover, from the variation of J in ϕ_b , we obtain the following Euler-Lagrange equation,

$$\begin{aligned} & -\phi_b |\mathbf{C}_b|^2 w_b^2 - \gamma_2 \nabla^2 \phi_b + \alpha_2 |\phi_b|^2 \phi_b - \beta_2 \phi_b - E \phi_b \\ = & -\gamma_2 \nabla^2 \phi_b + \alpha_2 |\phi_b|^2 \phi_b - \hat{\beta}_2 \phi_b - E \phi_b \\ = & 0, \end{aligned} \quad (42)$$

where $\hat{\beta}_2 = |\mathbf{C}_b|^2 w_b^2 + \beta_2$.

We have numerically solved such a system of equations.

Here, we define

$$v_a = \frac{|\phi_a|^2}{|\phi_a|^2 + |\phi_b|^2}$$

and

$$v_b = \frac{|\phi_b|^2}{|\phi_a|^2 + |\phi_b|^2}.$$

For the solutions v_a obtained for the values corresponding to a decreasing in temperature from T_1 to T_3 , $\beta_1(T_1) = 1050$, $\beta_1(T_2) = 500$ and $\beta_1(T_3) = 100$, please see figures 1, 2 and 3, respectively.

For the corresponding solutions v_b , please see figures 4, 5 and 6, respectively.

We may observe that for such solutions, we start at T_1 with a normal phase corresponding to $v_a \approx 1$ and $v_b \approx 0$, in Ω and, as the temperature decreases to T_3 , we obtain a super-conducting phase b , corresponding to $v_a \approx 0$ and $v_b \approx 1$, in Ω , as expected.

Finally, we highlight at T_3 and below T_3 , we may increase a lot w_b (which corresponds to increase $\hat{\beta}_2$) without qualitatively changing the solution, obtaining, for such a higher level of w_b , a much higher kinetics energy, the so-called super electric currents.

Here we present the software "SuperCond-July-2025" in MAT-LAB through which we have obtained such numerical results.

1. clear all
- m8=1000;
- d=1/m8;

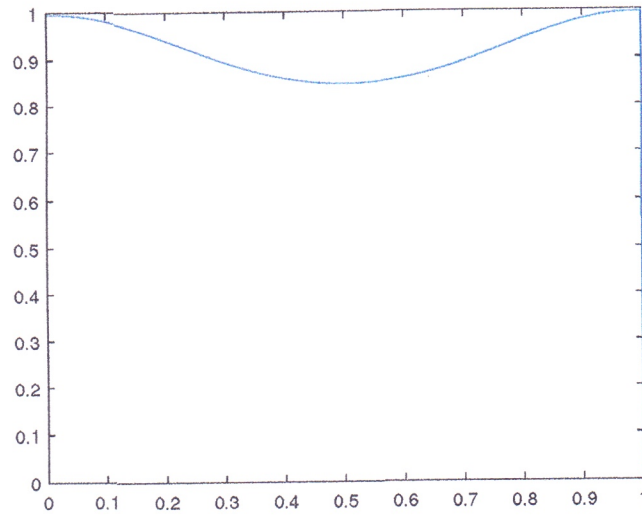


Figure 1: Graph for the normal phase $v_a(x)$ for the temperature T_1 .

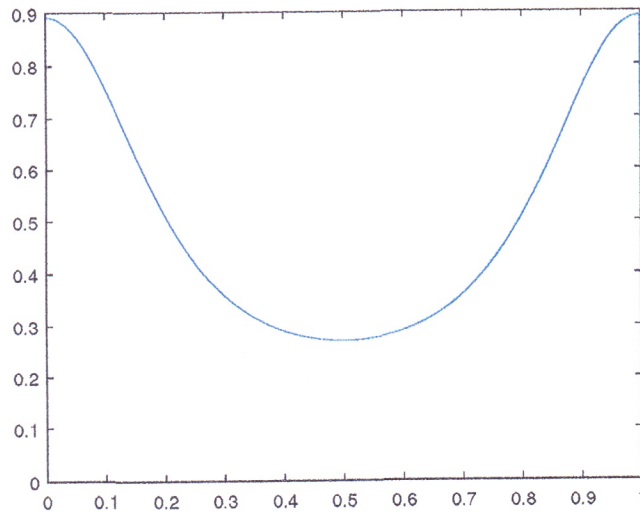


Figure 2: Graph for the normal phase $v_a(x)$ for the temperature $T_2 < T_1$.

K1=50;

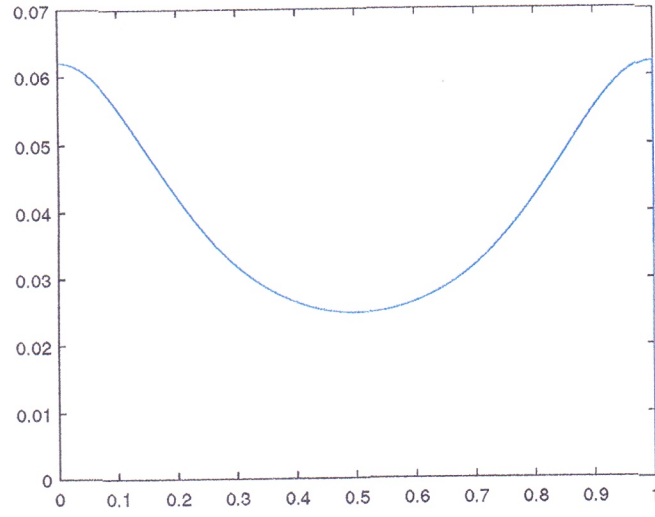


Figure 3: Graph for the normal phase $v_a(x)$ for the temperature $T_3 < T_2$.

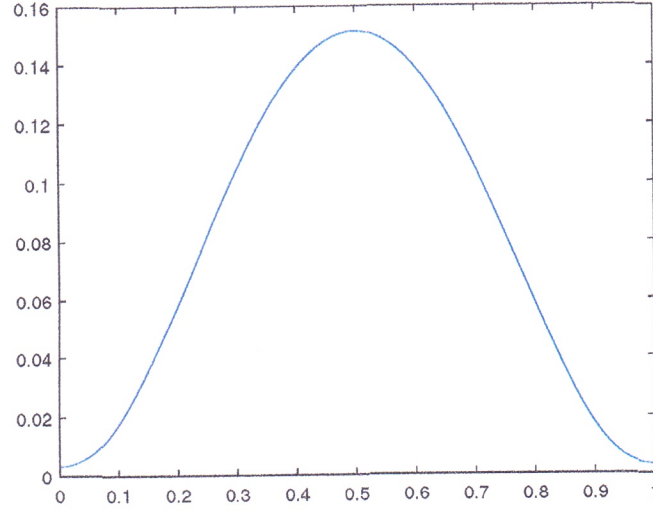


Figure 4: Graph for the super-conducting phase $v_b(x)$ for the temperature T_1 .

$$e1 = 10^{-10};$$

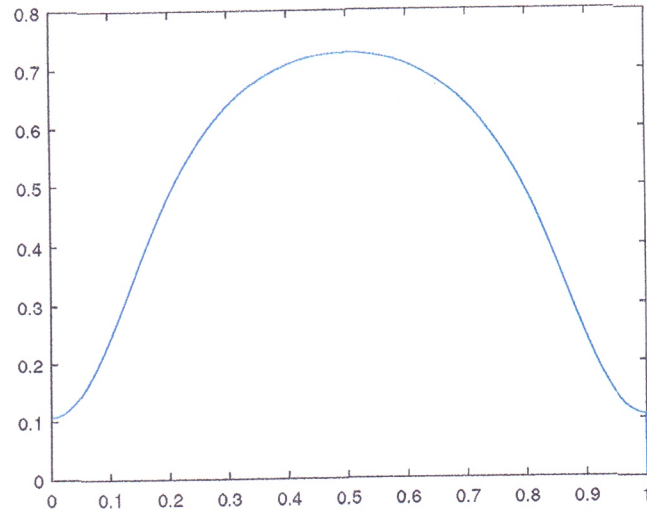


Figure 5: Graph for the super-conducting phase $v_b(x)$ for the temperature $T_2 < T_1$.

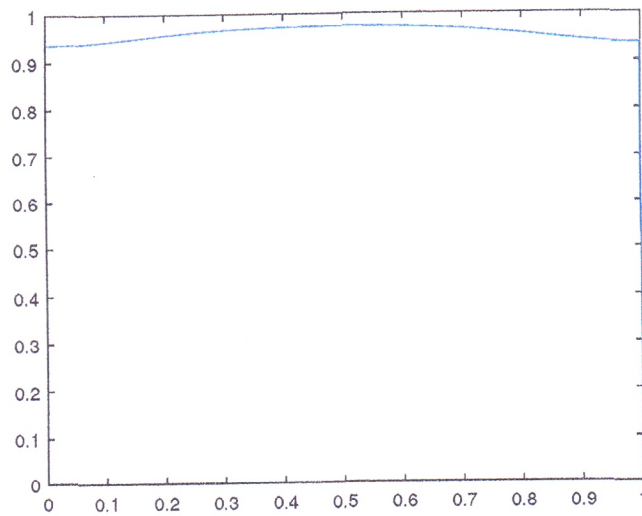


Figure 6: Graph for the super-conducting phase $v_b(x)$ for the temperature $T_3 < T_2$.

```
A1=1000;
A2=1.0;
B1=100;
B2=60.0;
for i=1:m8
ua(i,1)=10;
ub(i,1)=10;
end;
b12=1.0;
k=1;
while (b12 > 10-4) && (k < 9500)
k
k=k+1;
S=0;
for i=1:m8
S = S + ua(i,1)2 * d + ub(i,1)2 * d;
end;
S1=sqrt(S+e1);
i=1;
m12 = 2 + 3 * A1 * ua(i,1)2 * d2/S12 + K1 * d2 - B1 * d2;
m14 = 2 + 3 * A2 * ub(i,1)2 * d2/S12 + K1 * d2 - B2 * d2;
m50(i)=1/m12;
m70(i)=1/m14;
z1(i) = m50(i) * (ua(i,1) * d2/S1 + 2 * A1 * ua(i,1)3/(S12) * d2);
z2(i) = m70(i) * (ub(i,1) * d2/S1 + 2 * A2 * ub(i,1)3/(S12) * d2);
for i=2:m8-1
m12 = 2 + 3 * A1 * ua(i,1)2 * d2/(S12) + K1 * d2 - B1 * d2 - m50(i-1);
m14 = 2 + 3 * A2 * ub(i,1)2 * d2/(S12) + K1 * d2 - B2 * d2 - m70(i-1);
m50(i)=1/m12;
m70(i)=1/m14;
z1(i) = m50(i) * (ua(i,1) * d2/S1 + 2 * A1 * ua(i,1)3 * d2/S12 + z1(i-1));
```

```
z2(i) = m70(i) * (ub(i, 1) * d^2/S1 + 2 * A2 * ub(i, 1)^3 * d^2/S1^2 + z2(i - 1));
end;
u1(m8,1)=0;
u2(m8,1)=0;
for i=1:m8-1
    u1(m8-i,1)=m50(m8-i)*u1(m8-i+1,1)+z1(m8-i);
    u2(m8-i,1)=m70(m8-i)*u2(m8-i+1,1)+z2(m8-i);
end;
b12=max(abs(u1-ua))+max(abs(u2-ub));
ua=u1;
ub=u2;
end;
for i=1:m8
    va(i, 1) = ua(i, 1)^2/(ua(i, 1)^2 + ub(i, 1)^2 + e1);
    vb(i, 1) = ub(i, 1)^2/(ua(i, 1)^2 + ub(i, 1)^2 + e1);
    x1(i,1)=i*d;
end;
plot(x1,va);

*****
```

5 Conclusion

In this article, we have presented a functional which the variation leads to a system of partial differential equations intended to model a phase transition from a normal to a superconductive electronic state.

In a near future research we intend to address similar models in superfluidity.

References

- [1] J.F. Annet, Superconductivity, Superfluids and Condensates, 2nd edn. (Oxford Master Series in Condensed Matter Physics, Oxford University Press, Reprint, 2010)

- [2] L.D. Landau L.D. and E.M. Lifschits, A Course of Theoretical Physics, Vol. 5- Statistical Physics, part 1. (Butterworth-Heinemann, Elsevier, reprint 2008).
- [3] P. Constantin, C. Foias, Navier-Stokes equation, University of Chicago Press, Chicago, 1989.
- [4] C. De Lellis , E. Brué, D. Albritton, M. Colombo, V. Giri , M. Janisch and H. Kwon, Instability and Non-uniqueness for the 2D Euler Equations, after M. Vishik (AMS-219) Volume 219 in the series Annals of Mathematics Studies, Princeton University Press, New Jersey, 2024. <https://doi.org/10.1515/9780691257846>
- [5] M. Hamouda M, D. Han, C-Y. Jung , R. Temam, Boundary layers for the 3D primitive equations in a cube: the zero-mode, Journal of Applied Analysis and Computation. 2018; 8, No. 3, (873-889), DOI:10.11948/2018.873.
- [6] A. Giorgini, A. Miranville A, R. Temam, Uniqueness and regularity for the Navier-Stokes-Cahn-Hilliard system, SIAM J. of Mathematical Analysis (SIMA).2019; 51, 3, (2535-2574), <https://doi.org/10.1137/18M1223459>.
- [7] C. Foias, R.M. Rosa, R. Temam, Properties of stationary statistical solutions of the three-dimensional Navier-Stokes equations, J. of Dynamics and Differential Equations, Special issue in memory of George Sell. 2019; 31, 3, 1689-1741, <https://doi.org/10.1007/s10884-018-9719-2>.
- [8] R. Temam, Navier-Stokes Equations, AMS Chelsea, reprint (2001).
- [9] J.C. Tannehill, D.A. Anderson, R.H. Pletcher, *Computational Fluid Mechanics and Heat Transfer*, 2nd edn., Taylor and Francis, Philadelphia, PA, 1997.
- [10] J.C. Strikwerda, finite difference cchemes and partial differential equations, second edition Philadelphia, SIAM, 2004.
- [11] F.S. Botelho, "Approximate numerical procedures for the Navier-Stokes system through the generalized method of lines" Nonlinear Engineering, vol. 13, no. 1, 2024, pp. 20220376. <https://doi.org/10.1515/nleng-2022-0376>
- [12] F. Botelho, Existence of solution for the Ginzburg-Landau system, a related optimal control problem and its computation by the generalized method of lines, Applied Mathematics and Computation, 2012; 218 (11976-11989).

- [13] F.S. Botelho, An approximate proximal numerical procedure concerning the generalized method of lines. *Mathematics* 2022, 10(16), 2950; <https://doi.org/10.3390/math10162950>.
- [14] F. Botelho *Functional Analysis and Applied Optimization in Banach Spaces*, (Springer Switzerland, 2014).
- [15] F.S. Botelho, *Functional analysis, calculus of variations and numerical methods in physics and engineering*. Boca Raton, Florida, USA CRC Taylor and Francis; 2020.
- [16] F.S. Botelho, *Advanced Calculus and its Applications in Variational Quantum Mechanics and Relativity Theory*, CRC Taylor and Francis, Florida, 2021.
- [17] S. Sheng and Z. Shao, Concentration of mass in the pressureless limit of the Euler equations of one-dimensional compressible fluid flow, *Nonlinear Analysis, Real world applications*, 52, (2020), 103039. <https://doi.org/10.1016/j.nonrwa.2019.103039>
- [18] R.A. Adams and J.F. Fournier, *Sobolev Spaces*, 2nd edn. (Elsevier, New York, 2003).